

Crossover transition in the Fluctuation of Internet

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Gibrat's law predicts that the standard deviation of the growth rate of a node's degree is constant. On the other hand, the preferential attachment (PA) indicates that such standard deviation decreases with initial degree as a power law of exponent -0.5 . While both models have been applied to Internet modeling, this inconsistency requires the verification of their validation. Therefore we empirically study the fluctuation of Internet of three different time intervals (daily, monthly and yearly). We find a crossover transition from PA model to Gibrat's law, which has never been reported. Specifically Gibrat-law starts from small degree region and extends gradually with the increase of the observed period. We determine the validated periods for both models and find that the correlation between internal links has large contribution to the emergence of Gibrat law. These findings indicate neither PA nor Gibrat law is applicable to the actual Internet, which requires a more complete model theory.

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I. INTRODUCTION

The evolution of Internet has long been a hot subject since the dawn of complex network theory, due to its rich data, wide application and nontrivial properties [1–11]. While models based on either stochastic process or optimal strategy are continually proposed, an urgent question to be addressed is that which of them is really applicable or is validated to describe the actual evolution of Internet [1, 12]. This question concerns not only our understanding on the process of internet evolution, but also the possibility of our further goal of control and prediction of this large-scale system.

Most popular models of Internet base on the mechanism of preferential attachment (PA), which describes that the probability of a node to capture links is proportional to its current degree. It is considered to be essential for producing the power-law degree distribution [13–19]. While some evidences suggest PA is unapplicable for route-level Internet [1], other empirical studies based on mean-field approach support its validation for AS level (autonomous system level) and other types of networks [20–22].

Another approach to model Internet follows its statistical law instead of the detailed descriptions as PA [23]. The representative case is Gibrat law which has been introduced as the candidate of Internet model to characterize the dynamics of the constant appearance and disappearance of links and nodes [9, 24]. The traditional Gibrat law assumes that the growth rate of a variable such as population, the number of messages sent by a person or the degree of a node has an independent identically distributed (i.i.d) structure so that both its mean and standard deviation are independent of the initial value of the variable [25]. Although this assumption seems rejected by a variety of recent empirical studies [26–32], it succeeds in reproducing the exact power-law exponent of the degree distribution of Internet [9].

While the validation of both model is still controver-

sial, a more serious problem is that there exists an inconsistency even between themselves. As is indicated in Ref [32] and will be specified in section II in the present paper, the conditional standard deviation of degree growth rate of PA decays with initial degree as a power law of exponent -0.5 , which contradicts with Gibrat assumption. This raises the question that which model is more appropriate for describing the evolution of Internet not only at a mean-field level but also on a fluctuation aspect. Unfortunately previous empirical studies based on mean-field method [20–22] cannot distinguish PA and Gibrat law since both them cause the similar proportionate effect. While the fluctuation property may uncover some important nature of Internet, it has been rarely empirically studied. The main purpose of the present paper is to determine the actual fluctuation property of Internet topology and the scope of the validation of the two models, which is significant both theoretically and practically.

The paper is organized as follow. In section II we show the inconsistency between PA and Gibrat law by deriving the relation between the standard deviation of degree growth rate and initial degree. In section III we empirically study the fluctuation of Internet topology for three different time scale (daily, monthly and yearly). We find that the fluctuation of internet experience a crossover transition from PA model to Gibrat's law with the increase of the observed period. We determine the validated period for both PA and Gibrat's law respectively and discuss the possible cause of the emergence of Gibrat law. In section IV we draw the conclusion.

II. INCONSISTENCY BETWEEN GIBRAT LAW AND PA RULE

The proportionate effect described by Gibrat law can be formalized by the following random multiplicative pro-

cess:

$$k_i(t+1) = [1 + \varepsilon_i(t)]k_i(t), \quad (1)$$

where $k_i(t+1)$ and $k_i(t)$ are the degree of node i at time $t+1$ and t , and $\varepsilon_i(t)$ is a random process. The degree growth rate is defined as

$$r_i = \log \frac{k_i(t+1)}{k_i(t)}. \quad (2)$$

More generally, if we observe the system by interval Δt , the growth rate $\log(k_i(t+\Delta t)/k_i(t))$ is given by

$$r_i(\Delta t) \sim \sum_{j=t}^{t+\Delta t} \varepsilon_i(j). \quad (3)$$

The basic assumptions of Gibrat law are that ε_i is (1) independent of its initial degree and (2) uncorrelated in time [25]. The two assumptions indicate that the fluctuation property of degree growth, characterized by the standard deviation of $r_i(\Delta t)$ conditional to initial degree $k_0 = k_i(t)$ follows

$$\sigma_r(k_0) \sim \text{const.} \quad (4)$$

On the other hand the fluctuation of degree growth of PA behaves differently. The PA rule describes that the probability p_k of a new link to connect to a node relates to nothing else but the node's current degree, which is given by

$$p_k \propto k. \quad (5)$$

In other words the creation of links are uncorrelated with each other and the evolution of degree is a memoryless Markov process. By mean-field method, the evolution of the degree of a node is $dk/dt = g(t)k$, where $g(t)$ is usually a function related to the growth pattern of network size. Solving the equation we have

$$k(t) \propto \frac{G(t)}{G(\tau)}, \quad (6)$$

where τ is the birth time of the node and $G(t) = e^{\int g(t)dt}$. Now Let us denote random variable $X(t)$ as the number of new links connecting to a node at time t . Its i.i.d structure indicates that it follows the Binomial distribution, whose variance $\sigma_{X(t)}^2$ is proportional to $p_k(1-p_k)$. Considering $p_k \ll 1$, we have

$$\sigma_{X(t)}^2 \sim p_k. \quad (7)$$

The degree increment of the node from t to $t+\Delta t$ is $\Delta k = k(t+\Delta t) - k(t) = \sum_{i=t}^{t+\Delta t} X(i)$. According to the definition of the growth rate r , we have

$$r(\Delta t) \sim \frac{\sum_{i=t}^{t+\Delta t} X(i)}{k(t)}. \quad (8)$$

Reminding that the creation of links are uncorrelated in time, the conditional variance of $r(\Delta t)$ is

$$\sigma_r^2(k(t)) \sim \frac{1}{k(t)^2} \int_t^{t+\Delta t} \sigma_{X(i)}^2 di. \quad (9)$$

Substituting Eq(5)~Eq(7) to Eq(9) and replacing $k(t)$ with k_0 , we finally derive the fluctuation property of degree growth rate for PA

$$\sigma_r(k_0) \sim k_0^{-0.5}. \quad (10)$$

Note that Eq(10) is valid for other events such as rewiring and link deletion as long as they do not break the memoryless property.

Eq(4) and Eq(10) indicate a basic contradiction between PA and Gibrat law even though both of them are reported to be validated at mean-field level. Our question is which model is closer to the reality and what is the real fluctuation property of Internet on earth. On the other hand PA and Gibrat law share common stationary property in the sense that both the scaling properties of Eq(4) and Eq(10) are independent of the observed period Δt . As will be presented in the next section, neither of the models can totally characterize the real fluctuation but is validated for two different periods. With the increase of the periods, the fluctuation pattern changes gradually, which contrasts to the stationary property of both models.

III. EMPIRICAL RESULT AND CORRELATION ANALYSIS

In this section we will empirically study the fluctuation of degree growth rate of Internet and determine the periods Δt , for which PA and Gibrat law are validated respectively. In addition we will briefly discuss the origin of the emergence of Gibrat law.

Our empirical data come from the Oregon Route Views project [33]. They include snapshots of three different time scales, i.e. daily(30 days: 2006/09/01 ~ 2006/09/30), monthly(36 months: 2005/01 ~ 2007/12) and yearly(15 years: 1998 ~ 2012). The original data are collected in the form of Border Gateway Protocol routing tables, from which an Internet graph can be constructed. As usual, each node represents a specific AS while each edge is the logical link between the inter-connected ASes, so that we obtain a network of size of $O(10^5)$ nodes and of an almost constant average degree about 4.5. The topological properties that we measured are stationary for all the three time scale and are consistent with previous empirical studies [4, 11]. The degree distribution is power law as $p(k) \sim k^{-\alpha}$ with exponent $\alpha \approx 2.1$. We also check the dynamics of the preferential attachment as done in Ref [20]. We find for all the three time scale, the linear PA $\Delta k \sim k$ is always valid.

The fluctuation property $\sigma_r(k_0)$ can be calculated by

$$\sigma_r(k_0) \equiv \sqrt{\langle r^2(\Delta t) \rangle - \langle r(\Delta t) \rangle^2}, \quad (11)$$

where $\langle \rangle$ represents the average taken for the same k_0 and the observed period Δt can be one year, one month or one day in the present paper. In Fig. 1(a), we plot the conditional mean of r for different periods. All of them are around constant zero, which is independent of the initial degree. However the conditional deviation of the three periods display different behaviors, as is shown in Fig. 1(b) ~ Fig. 1(d). For daily fluctuation, $\sigma_r(k_0)$ decays as power law with exponent about -0.5 , which coincides with the prediction of PA rule. For monthly fluctuation, the small-degree region of $k < 10$ becomes flat while the rest of region remains unchanged. With the increase of Δt , the flat area extends gradually and Gibrat law becomes dominated for a large region of $k < 300$ for yearly fluctuation. These results indicate a crossover transition from PA to Gibrat phase, which clearly rejects the stationarity of the fluctuation. Therefore neither PA nor Gibrat law can characterize the overall fluctuation property of Internet. They validate only for a specific period. For short period PA matches while for long period Gibrat law takes over. Note that our finding is different from those of human dynamics and firm growth, where a single universal scaling law is reported for the whole conditional deviation [26, 28, 32].

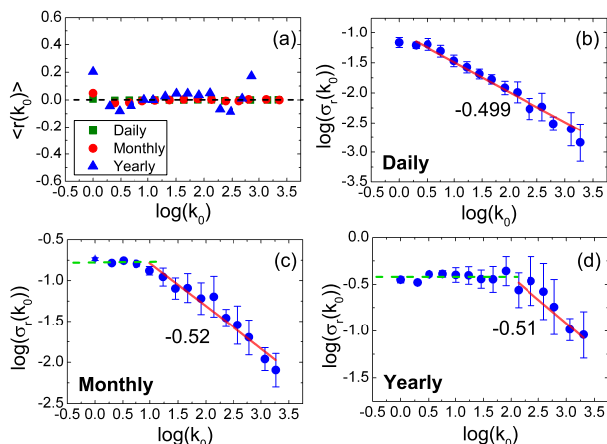


FIG. 1: Conditional average and conditional standard deviation of the degree growth rate versus initial degree: (a) the conditional average for three periods. All of them are independent of initial degree k_0 and stay around constant 0. (b) the conditional standard deviation for daily data. It decreases with k_0 as power law of exponents -0.5 , as predicted by PA. (c) the conditional deviation for monthly data. The small-degree region of $k_0 < 10$ becomes flat compared to daily data, while the rest of region remains unchanged. (d) the conditional deviation for yearly data. Gibrat law dominates for a large range of $k_0 < 300$. All the data are logarithmic binned and are plotted on a log-log scale. Red lines represent fitted results.

To better understand the scope of the application of the two classical models, we need to determine their validated periods Δt_v . For PA, we find the corresponding Δt_v is no more than several-day magnitude and we can

affirm that PA is always valid for $\Delta t_v < 1\text{day}$ as is indicated in Fig. 1(b). For Gibrat law, Δt_v corresponds to when the correlation coefficient of $\sigma_r(k_0)$ and k_0 is zero. Therefore we study the relation between the correlation coefficient and the observed period Δt by using monthly data. Specifically, for a particular Δt we calculate the correlation coefficients $C_{\sigma,k}(t, \Delta t)$ for all $t \in \{1, 2, 3, \dots, 36 - \Delta t\}$ and average them so that

$$C(\Delta t) = \frac{\sum_{t=1}^{36-\Delta t} C_{\sigma,k}(t, \Delta t)}{36 - \Delta t}. \quad (12)$$

We consider $C(\Delta t)$ characterize the general correlation coefficient of $\sigma_r(k_0)$ and k_0 for the observed interval Δt . We calculate Eq(12) for $\Delta t \in \{1, 2, \dots, 28\}$ and present its absolute value in Fig. 2 [34]. We find that despite large deviation for the results of $\Delta t < 4$ (not plotted), the main body of $|C(\Delta t \geq 4)|$ displays a linear decay which is fitted as

$$|C(\Delta t)| \approx -0.005 \times \Delta t + 0.28 \quad (13)$$

Then let $|C(\Delta t)| = 0$, we can evaluate $\Delta t_v = 56\text{months} \approx 4.6\text{years}$. Therefore the Gibrat law is expected to be totally valid for at least 4.6-year period. Note that Δt_v of Gibrat law estimated by using yearly data gives the similar result even though the quality of the fitting is poorer due to much smaller length of both t and Δt .

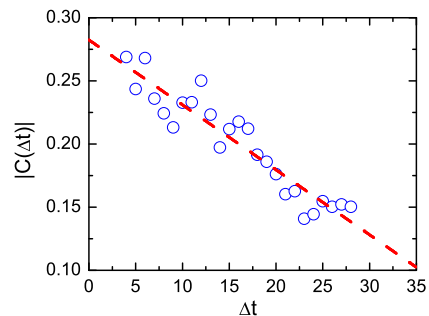


FIG. 2: Absolute value of the average correlation between k_0 and $\sigma_r(k_0)$ versus observed period Δt . It follows approximately a linear decrease, which is fitted by the red dashed line as $|C(\Delta t)| \approx -0.005 \times \Delta t + 0.28$.

The crossover transition indicates that there are some underlying mechanism that give rise to the emergence of Gibrat law. Reminding that memoryless and independent creation of links can only cause a power-law decay with an exponent of -0.5 of $\sigma_r(k_0)$, Gibrat law with constant conditional standard deviation probably indicates the existence of strong correlation in the evolution of Internet. Indeed studies on population growth and human communication dynamics demonstrated that correlation could lower the related power-law exponent [27, 32, 35]. This speculation can be confirmed by reshuffling the creation of links for yearly data. In specific, we change randomly the order of the creation of links while maintain

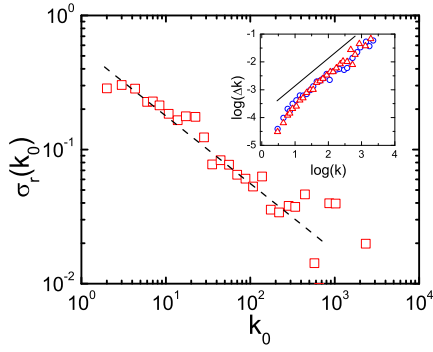


FIG. 3: The conditional standard deviation for the reshuffled yearly data. It follows a power-law decay of exponent about -0.5 , as is indicated by the black dashed line. The result contrasts to that of the original yearly data but is consistent with that of PA and daily data. Inset: The empirical result of the proportionate effect before (blue circle) and after (red triangle) the reshuffling operation. The statistical analysis is based on mean-field treatment as was done in previous studies [20–22]. The black solid line is of slope 1, which is a guide for eyes.

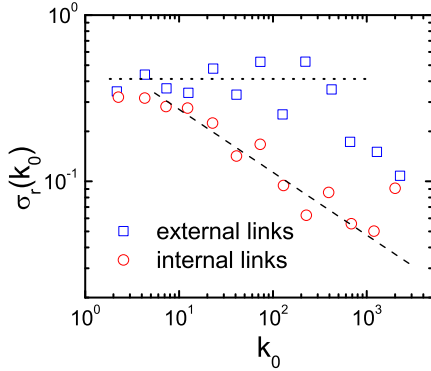


FIG. 4: The conditional standard deviation for reshuffling the creation of external (blue square) and internal (red circle) links. $\sigma_r(k_0)$ after reshuffling the external links appears no significant difference from that of the original yearly data. However $\sigma_r(k_0)$ after reshuffling the internal links decays with power-law exponent about -0.43 . The dotted line is horizontal while the dashed one is the fit line.

the topology of first year (1998) and last year (2012). We first check whether the reshuffling operation changes the basic evolution mechanism, i.e. the proportionate effect. Surprisingly, the proportionate effect $\Delta k \sim k$ maintains as before (inset of Fig. 3), but the fluctuation pattern of the degree growth rate changes from a constant value to a power-law decay of exponent -0.5 , which is exactly the behavior of PA and daily data (Fig. 3). This is a direct evidence for the existence of the correlation and its contribution to Gibrat law. Indeed the reshuffling process

does not change PA at mean-field level at all but only destroys any possible correlation between the creation of links. Therefore we draw the conclusion that correlation is the essential ingredient responsible for the emergence of Gibrat law. The crossover transition thus indicates that such a correlation occurs first at small-degree nodes and spreads to large-degree nodes gradually. To further identify the origin of Gibrat law, we separate the external links (links created between new-coming node and old existing node) from internal links (links created between existing old nodes) and reshuffle one while maintain the other. As shown in Fig. 4, reshuffling the external links has little effect on the fluctuation pattern. On the other hand reshuffling the internal links causes a clear power law decay of the conditional standard deviation with exponents about -0.43 . Therefore we conclude that the major part of the correlation comes from the internal links, which has more contribution to the emergence of Gibrat law.

IV. CONCLUSION

We have shown the inconsistency between PA and Gibrat law and determine their scope of application to Internet. By analyzing the conditional standard deviation of the degree growth rate, we find that the actual fluctuation of Internet exhibits a crossover transition from PA to Gibrat law with the increase of the observed period. We have determined that the scope of the validation is about several-day magnitude of period for PA while 4.6-year of period for Gibrat law. We briefly study the origin of the emergence of Gibrat law and find it most related to the correlation between the internal links.

There has been an argument that whether the construction of Internet is governed by the randomness of self-organized nature or highly designed order of engineered nature [1]. Although self-organized system does not rule out the possibility of correlation, the strong correlation found in the evolution of Internet consists with the designed order of the engineered intuition. The present empirical results indicate that purely random description based on mean-field approach, which ignores correlation, might match short-term (daily) Internet fluctuation, but is very insufficient to characterize the long-term (yearly) evolution. The crossover paradigm of the dynamical fluctuation provides a test that any future model should pass. Therefore the consideration of memory effect [37] as well as how such effect works is critical for a complete Internet model theory.

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